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MATHEMATICAL PROBLEMS IN THE INTEGRAL-TRANSFORMATION METHOD OF DYNAMIC CRACK*

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Abstract: *In the investigation on fracture mechanics, the potential function was introduced, and the moving differential equation was constructed. By making Laplace and Fourier transformation as well as sine and cosine transformation to moving differential equations and various responses, the dual equation which is constructed from boundary conditions lastly was solved. This method of investigating dynamic crack has become a more systematic one that is used widely. Some problems are encountered when the dynamic crack is studied. After the large investigation on the problems, it is discovered that during the process of mathematic derivation, the method is short of precision, and the derived results in this method are accidental and have no credibility. A model for example is taken to explain the problems existing in initial deriving process of the integral-transformation method of dynamic crack.*

Key words: potential function; integral transform; dynamic crack; dual equation

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Introduction

In recent years, the investigation on dynamic damage and crack of various materials is very popular. The integral-transformation method has become a more systematic one. By use of this method, many monographs on the research of isotropy material have been published. Shindo Y. *et al*.[4] studied the scattering of vertical incidence transient longitudinal wave on finite crack in piezoelectric medium under a uniform electric field in the integral-transformation method. Li S. *et al*.[2,3] investigated the anti-plane transient response of semi-infinite spread crack in piezoelectric medium. Chen Z.T. *et al*.[4,5] studied anti-plane impact problems of finite crack

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and anti-plane dynamic crack in piezoelectric materials by means of the integral-transformation. The authors of this paper decide to study dynamic fracture problems of piezoelectric medium in this method as well. But during the application, the authors discover that the integration of the core function in the integral equation is divergent. Therefore, the authors deduce this method once again, and find that the former mathematic derivation is imprecise.

Now we only take a model for example to point put where the problem appears.

1 The Problem of Finite Type I Crack Problem on Infinite Plane Under Impact Load

Assuming infinite plane as xoy plane, ignoring the effect of volume force, only considering the effect of inertia force, we introduce Lame potential functions $\varphi(x,y,t)$, $\psi(x,y,t)$, and give the following displacement functions:

$$u_x = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \quad u_y = \frac{\partial\varphi}{\partial y} + \frac{\partial\psi}{\partial x}. \tag{1}$$

According to Hooke's law, the stress components are

$$\begin{cases} \sigma_{xx} = \lambda\nabla^2\varphi + 2\mu\left(\dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial x\partial y}\right), \\ \sigma_{yy} = \lambda\nabla^2\varphi + 2\mu\left(\dfrac{\partial^2\varphi}{\partial y^2} + \dfrac{\partial^2\psi}{\partial x\partial y}\right), \\ \sigma_{xy} = \mu\left(2\dfrac{\partial^2\varphi}{\partial x\partial y} - \dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2}\right), \end{cases} \tag{2}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, then the wave equations described with potential functions $\varphi(x,y,t)$ and $\psi(x,y,t)$ are

$$\nabla^2\varphi = \frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2}, \quad \nabla^2\psi = \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2}, \tag{3}$$

where c_1 and c_2 are respectively longitudinal wave velocity and transverse wave velocity.

For Type I crack problem, we assume that (i) The crack is on x axis, length $L = 2a$; (ii) The center is at grid origin; (iii) The medium is not affected by the load at infinite distance, but affected by the impact load on both crack surfaces. Because of symmetry, it is only necessary to study 1/4 plane. Its boundary conditions can be written as

$$\begin{cases} \sigma_{yy}(x,0,t) = -\sigma_0 \mathrm{H}(t) & (0 \leqslant x < a, t > 0), \\ u_y(x,0,t) = 0 & (a < x < \infty, t > 0), \\ \sigma_{xy}(x,0,t) = 0 & (0 \leqslant x < \infty, t > 0), \\ \sigma_{ij}(x,y,t) \to 0, \quad \sqrt{x^2+y^2} \to \infty, & t > 0, \end{cases} \tag{4}$$

where $H(t)$ is Heaviside unit jump function. Assuming that the body is still initially, then the initial condition is

$$u_j(x,y,0) = \dot{u}_j(x,y,0) = 0. \tag{5}$$

Making Laplace transformation to Eq.(3), and considering initial condition, we derive the results as follows:

$$\nabla^2\varphi^* = \frac{1}{c_1^2}p^2\varphi^*, \quad \nabla^2\psi^* = \frac{1}{c_2^2}p^2\psi^*. \tag{6}$$

The potential functions have following properties:

$$\varphi(x,y,t) = \varphi(-x,y,t), \quad \psi(x,y,t) = -\psi(-x,y,t).$$

Considering this point, we make cosine and sine Fourier transformation about " x " to Eq.(6), respectively, so that

$$\frac{\partial^2 \bar{\varphi}^*}{\partial y^2} - s^2 \bar{\varphi}^* = \frac{1}{c_1^2} p^2 \bar{\varphi}^*, \quad \frac{\partial^2 \bar{\psi}^*}{\partial y^2} - s^2 \bar{\psi}^* = \frac{1}{c_1^2} p^2 \bar{\psi}^*. \tag{7}$$

According to Eq.(7), $\bar{\varphi}^*$ and $\bar{\psi}^*$ that satisfy the infinite distance boundary conditions are

$$\bar{\varphi}^*(s,y,p) = A_1(s,p)\exp(-\gamma_1 y),$$

$$\bar{\psi}^*(s,y,p) = A_2(s,p)\exp(-\gamma_2 y),$$

where $\gamma_j = \sqrt{s^2 + (p/c_j)^2}$, and the undetermined function $A_j(s,p)$ can be determined with else boundary conditions.

In Eq. (7), after solving the primitive function of Fourier image function, we have following results:

$$\begin{cases} \varphi^*(x,y,p) = \dfrac{2}{\pi}\displaystyle\int_0^\infty A_1(s,p)\exp(-\gamma_1 y)\cos(sx)\mathrm{d}s, \\ \psi^*(x,y,p) = \dfrac{2}{\pi}\displaystyle\int_0^\infty A_2(s,p)\exp(-\gamma_2 y)\sin(sx)\mathrm{d}s. \end{cases} \tag{8}$$

Making Laplace transformation to the boundary conditions, we get

$$\sigma_{yy}^*(x,0,p) = -\sigma_0/p, \quad \sigma_{xy}^*(x,0,p) = 0 \qquad (0 < x < a),$$

$$u_y^*(x,0,p) = 0, \quad \sigma_{xy}^*(x,0,p) = 0 \qquad (x > a).$$

Making Laplace transformation to Eq.(1) and Eq.(2), and substituting Eq.(8) into them, respectively, we can obtain the displacement and stress components in the Laplace transformation domain.

According to boundary conditions $\sigma_{xy}^*(x,0,p) = 0$, $0 \leqslant x < \infty$, we can obtain $2\gamma_1 s A_1(s,p) = -(s^2 + \gamma_2^2)A_2(s,p)$, assuming that

$$\begin{cases} A_1(s,p) = -\dfrac{(s^2 + \gamma_2^2)}{2\gamma_1} A(s,p), \\ A_2(s,p) = sA(s,p). \end{cases} \tag{9}$$

According to boundary conditions $u_y^*(x,0,p) = 0$, $(x > a)$ and $\sigma_{yy}^*(x,0,p) = -\sigma_0/p$ $(0 < x < a)$, we have the following result:

$$\begin{cases} \displaystyle\int_0^\infty A(s,p)\cos(sx)\mathrm{d}s = 0 \qquad (x > 0), \\ \displaystyle\int_0^\infty sf(s,p)A(s,p)\cos(sx)\mathrm{d}s = \dfrac{\pi\sigma_0 c_2^2}{2\mu p^3(1-k^2)} \qquad (0 < x < a), \end{cases} \tag{10}$$

where $f(s,p) = (2\gamma_1 s(1-k^2))^{-1}\{[s^2 + \gamma_2^2]^2 - 4s^2\gamma_1\gamma_2\}(c_2/p)^2$, $k = \sqrt{c_2/c_1}$.

Thus the problem can be summarized as the problem of determining $A(s,p)$ that satisfies the dual integral equation (10), assuming that

$$A(s,p) = \frac{\pi\sigma_0 a^2 c_2^2}{2\mu p^3(1-k^2)}\int_0^1 \sqrt{\xi}\Phi_1^*(\xi,p)J_0(sa\xi)\mathrm{d}\xi, \tag{11}$$

and then substituting Eq.(11) into Eq.(10), we can find that the first equation of Eq.(10) is satisfied naturally and the second equation is transformed to Fredholm integral equation

$$\Phi_1^*(\xi,p) + \int_0^1 [K_1(\xi,\eta,p)\Phi_1^*(\eta,p)]d\eta = \sigma_0\sqrt{\xi}, \tag{12}$$

where the core of the integral equation is

$$K_1(\xi,\eta,p) = \sqrt{\xi\eta}\int_0^\infty s[f(s/a,p) - 1]J_0(sa\xi)J_0(sa\eta)ds.$$

After solving the integral equation (12), we can further calculate the stress intensity factor.

2 Two Mathematic Problems in the Above Method

The authors find that there are two problems existing in the above method as follows:

1) The core of the integral equation

$K_1(\xi,\eta,p) = \sqrt{\xi\eta}\int_0^\infty s[f(s/a,p) - 1]J_0(sa\xi)J_0(sa\eta)ds$ does not exist. The reason is that when "s" is quite large, the following equation comes into being

$$J_n(s) \approx \sqrt{\frac{2}{\pi s}}\cos\left(s - \frac{\pi}{4} - \frac{n\pi}{2}\right), \tag{13}$$

and γ_j can approximately be expressed as

$$\gamma_j = s + \frac{1}{2s}\left(\frac{p}{c_j}\right)^2 + o(s^{-3}), \tag{14}$$

$$\lim_{s\to\infty} f(s,p) = 1 + k^2, \tag{15}$$

$$\lim_{s\to\infty}\left\{s\left[f\left(\frac{s}{a},p\right) - 1\right]J_0(sa\xi)J_0(sa\eta)\right\} =$$

$$\lim_{s\to\infty}\left\{sk^2\sqrt{\frac{2}{\pi sa\xi}}\cos\left(sa\xi - \frac{\pi}{4}\right)\sqrt{\frac{2}{\pi sa\eta}}\cos\left(sa\eta - \frac{\pi}{4}\right)\right\} =$$

$$\lim_{s\to\infty}\left\{\frac{2k^2}{a\pi}\sqrt{\frac{1}{\xi\eta}}\cos\left(sa\xi - \frac{\pi}{4}\right)\cos\left(sa\eta - \frac{\pi}{4}\right)\right\}.$$

It can be seen that the integrated function of the core function does not converge, i.e., $\lim_{s\to\infty} K_1(\xi,\eta,p) \to \infty$. The core function does not exist, so the integral equation (12) cannot be solved by use of this method.

2) During the process of deriving from Eq. (10) to Eq. (12), some properties of Bessel function are used.

Substituting $\cos(sx) = \sqrt{\pi xs/2}\,J_{-1/2}(sx)$ and Eq. (11) into the dual integral equation (10), we can obtain following dual integral equation:

$$\begin{cases}\int_0^\infty\left[\int_0^1\sqrt{\xi}\Phi_1^*(\xi,p)J_0(sa\xi)d\xi\right](\sqrt{s}J_{-1/2}(sx))ds = 0 & (x > 0),\\ \int_0^\infty s^{3/2}f(s,p)\left[a^2\int_0^1\sqrt{\xi}\Phi_1^*(\xi,p)J_0(sa\xi)d\xi\right]J_{-1/2}(sx)ds = 1 & (0 < x < a).\end{cases} \tag{16}$$

According to the Bessel function differential formula $d[z^{-\nu}J_\nu(z)]/dz = -z^{-\nu}J_{\nu+1}(z)$, making integration to $\left[\int_0^1\sqrt{\xi}\Phi_1^*(\xi,p)J_0(sa\xi)d\xi\right]$ of Eq. (13) in part, we can get following result:

$$\int_0^1\sqrt{\xi}\Phi_1^*(\xi,p)J_0(sa\xi)d\xi = \Phi_1^*(1,p)J_{-1}sa + \int_0^1\xi J_{-1}(sa\xi)d\left(\frac{\Phi_1^*(\xi,p)}{\sqrt{\xi}}\right). \tag{17}$$

Substituting Eq. (17) into Eq. (16), and using the discontinuous integration formulae of Bessel function, we can attain

$$\begin{cases} \int_0^\infty J_\lambda(r\xi) J_\mu(b\xi)\xi^{1+\mu-\lambda}\mathrm{d}\xi = 0 & (0 < r < b), \\ \int_0^\infty J_\lambda(r\xi) J_\mu(b\xi)\xi^{1+\mu-\lambda}\mathrm{d}\xi = \dfrac{b^\mu(r^2 - b^2)^{\lambda-\mu-1}}{2^{\lambda-\mu-1}r^\lambda\Gamma(\lambda - \mu)} & (0 < b < r), \end{cases} \tag{18}$$

where $\Gamma(z)$ is Γ function, which requires $\lambda > \mu > -1$, and when $r = b$, $\lambda > \mu > 0$. But when substituting Eq.(17) into Eq.(16), we attain that those who are correspondent with Eq.(18) are $\lambda = -1/2, \mu = -1$, but not $\mu > -1$, and when $r = b$, they do not satisfy $\lambda > \mu > 0$.

Further more, in calculating, we also apply the Abel type integral equation and its inversion formula

$$\begin{cases} f(r) = \dfrac{2^\alpha}{\Gamma(1-\alpha)} r^{-\nu} \int_0^r \dfrac{\mathrm{d}}{\mathrm{d}t}[r^{\nu+\alpha+1}\phi(t)] \dfrac{\mathrm{d}t}{(r^2 - t^2)^{1-\alpha}} & (0 < x < 1), \\ \phi(t) = \dfrac{2^{1-\alpha}}{\Gamma(\alpha)} r^{1-\nu-\alpha} \int_0^t \dfrac{r^{1+\nu} f(r)\mathrm{d}r}{(t^2 - r^2)^{1-\alpha}} & (0 < \xi < 1). \end{cases} \tag{19}$$

Eq.(19) demands $\lambda > \alpha$ and $0 < \alpha < 1$. But when substituting Eq.(17) into Eq.(16), we attain that those who are correspondent with Eq.(19) are $\lambda = -1/2$, $\alpha = 1/2$, which is just $\lambda = \alpha$ but not $\lambda > \alpha$.

After above farraginous derivation, the integral equation (12) may be obtained constrainedly. It is impossible for the results obtained by this way to have credibility and validity. And that Eq.(12) cannot be solved is certain.

3 Conclusion

As indicated above, we can see that some problems exist in "The integral transformation method of dynamic crack". It is imprecise in the mathematical derivation.

References:

[1] Shindo Y, Katsure H, Yan W. Dynamic stress intensity factor of a cracked dielectric medium in a uniform electric field[J]. *Acta Mech*, 1996:**117**(1-4):1 - 10.

[2] Li S, Mataga P A. Dynamic crack propagation in piezoelectric materials—Part Ⅰ: Electrode solution[J]. *J Mech Phys Solids*, 1996, **44**(11):1799 - 1830.

[3] Li S, Mataga P A. Dynamic crack propagation in piezoelectric materials—Part Ⅱ: Vacuum solution[J]. *J Mech Phys Solids*, 1996, **44**(11):1831 - 1866.

[4] Chen Z T, Yu S W. Anti-plane Yoffe crack problem in piezoelectric materials[J]. *Int J Fractrue*, 1997, **84**(3):L41 - L45.

[5] Chen Z T, Karihaloo B L. Dynamic response of a cracked piezoelectric ceramic under arbitrary electro-mechanical impact[J]. *Int J Solids Struct*, 1999, **36**(3):5125 - 5133.

[6] WANG Zhu-xi, GUO Dun-ren. *Spcial Function Introduction*[M]. Beijing: Science Press, 1979. (in Chinese)

[7] FAN Tian-you. *Fracture Dynamics Introduction*[M]. Beijing: Beijing Science and Technology University Press, 1990. (in Chinese)

[8] Busbridge I W. *Proc London Math Soc*, 1838, **44**:115 - 124.

[9] HOU Mi-shan, MEI Fu-liang. Problems of antiplane strain of electrically permeable cracks between bonded dissimilar piezoelectric materials[J]. *Chinese Science Bulletin*, 1998, **43**(4):341.

[10] Freund L B. *Dynamic Fracture Mechanics*[M]. Cambridge: Cambrideg Press, 1990.